# ON UNIFORMLY EXACT EQUATIONS OF THE PLANE LAMINAR BOUNDARY LAYER FOR A BODY WITH A SHARP LOCAL CHANGE IN THE CURVATURE OF ITS PROFILE

PMM Vol. 31, No. 5, 1967, pp. 804-811

B. M. BULAKH
(Leningrad)

(Received December 12, 1966)

The approximate equations of laminar motion of gas in a plane boundary layer, the Prandtl equations, are derived from the Navier-Stokes equations by neglecting second-order terms. In estimating terms in the equations it is assumed that the radius of curvature of the profile of the body in the stream is of the order of its length, and that the Reynolds number $R$ of the problem satisfies $R \gg 1$. More precisely, Prandtl's equations may be regarded as the equations for the leading terms in an asymptotic expansion of the solution of the Navier-Stokes equations in a series in powers of $\varepsilon = R^{-1/2}$ as $\varepsilon \to 0$; Prandtl's equations being accurate to within a factor $1 + O(\epsilon)$. The limitation mentioned on the curvature $\varkappa$ of the profile is not essential in the asymptotic theory, where $\varkappa$ need only be bounded for $\epsilon \to 0$, but is important for application of the equations to small but finite $\epsilon$. We consider below the case when the radius of curvature of the body profile is at some point small in comparison with the length of the body. Here the Prandtl equations may be too crude for small but finite $\epsilon$, and the solution of the boundary-layer problem is regarded as depending upon the two parameters $\epsilon$ and $\delta$, where $\delta$ is the minimum radius of curvature of the profile. In the paper an asymptotic theory is developed for the laminar motion of gas with $\epsilon \to 0$, $\delta \to 0$ in the vicinity of points of the profile where a sharp change of curvature occurs, in the case when the gas stream outside the boundary layer is supersonic. The equations of gas motion are derived with accuracy to within a factor $1 + O(\epsilon)$ for different rates of approach of $\epsilon$ and $\delta$ to zero. Possible means of solving these equations for small but finite $\epsilon$ and $\delta$ are discussed.

**1.** We consider a profile whose curvature $\varkappa$ is a continuous function of the coordinate $s$ measured along the profile (see Fig. 1) from the point $O$, where $\varkappa$ attains its greatest value $\varkappa_{max}$, and the radius of curvature correspondingly its minimum value $\delta = (\varkappa_{max})^{-1}$. We take the distance from point $A$ to point $O$, measured along the profile, as the characteristic length $\ell_0$, and will refer all lengths to $\ell_0$. We assume that the profile is such that

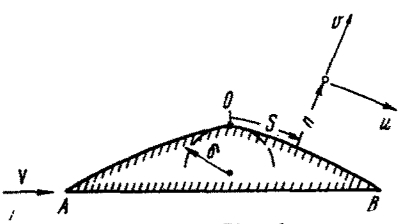


Fig. 1

$$\int_{-\delta}^{\delta} \varkappa \, ds \gg \int_{s-\delta}^{s+\delta} \varkappa \, ds$$

that is, the curvature of the profile in the vicinity of point $O$ is much greater than in the vicinity of all other points (for which $s \neq 0$). With an eye to subsequent aims, we embed the profile under consideration in a family of profiles whose shape depends parametrically upon $\delta$ : the curvature of the profiles $\varkappa = \varkappa(s, \delta)$ is taken to be continuous and to satisfy the condition

$$\lim_{\varepsilon,\,\delta\to 0} \int_{s-\delta}^{s+\delta} \varkappa ds = \begin{cases} 0, & s \neq 0 \\ c, & s = 0 \end{cases} \qquad (0 < c < 2) \tag{1.1}$$

The equation of this family of profiles is not required in what follows, and will not be derived. We simply note that the schematic form of the profile in the vicinity of point $O$ can, for $\delta \to 0$, be described by a corner rounded off by a circle of radius $\delta$. It follows from condition (1.1) that in the vicinity of point $O$, $\varkappa$ can be represented by

$$\varkappa = K(S, \delta)\,\delta^{-1}, \qquad S = s\delta^{-1}, \qquad K(S, \delta) = \frac{\varkappa}{\varkappa_{\max}} \leqslant 1 \tag{1.2}$$

The function $K(S, \delta)$ has the property

$$\int_{-1}^{1} K(S, \delta)\, dS \to c, \quad \delta \to 0 \qquad \text{or} \quad 0 < K(S, \delta) \leqslant 1 \quad \text{for } -1 < S < 1, \quad \delta \to 0$$

**2.** Now let a uniform stream of viscous perfect gas flow past a body having this profile. In the system of coordinates used in boundary-layer theory, where $s$ is measured along the profile and $n$ normal to it, the Eqs. of continuity, momentum, energy and state for a gas have the form

$$(\rho u)_s + [(1 + \varkappa n)\rho v]_n = 0 \tag{2.1}$$

$$\begin{aligned} &\frac{1}{\varepsilon^2}\left[\rho\left(\frac{u u_s}{1+\varkappa n} + v u_n + \frac{\varkappa}{1+\varkappa n} uv\right) + \frac{p_s}{1+\varkappa n}\right] = \\ &= \left[\mu\left(u_n + \frac{v_s - \varkappa u}{1+\varkappa n}\right)\right]_n + \frac{2}{1+\varkappa n}\left[\mu\left(\frac{u_s + \varkappa v}{1+\varkappa n}\right)\right]_s + \\ &+ \mu\frac{2\varkappa}{1+\varkappa n}\left(u_n + \frac{v_s - \varkappa u}{1+\varkappa n}\right) + \frac{1}{1+\varkappa n}\left[\lambda\left(\frac{u_s + \varkappa v}{1+\varkappa n} + v_n\right)\right]_s \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\frac{1}{\varepsilon^2}\left[\rho\left(\frac{u v_s}{1+\varkappa n} + v v_n - \frac{\varkappa}{1+\varkappa n} u^2\right) + p_n\right] = 2(\mu v_n)_n + \\ &+ \frac{1}{1+\varkappa n}\left[\mu\left(u_n + \frac{v_s - \varkappa u}{1+\varkappa n}\right)\right]_s + 2\mu\left(v_n - \frac{u_s + \varkappa v}{1+\varkappa n}\right)\frac{\varkappa}{1+\varkappa n} + \left[\lambda\left(\frac{u_s + \varkappa v}{1+\varkappa n} + v_n\right)\right]_n \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\frac{1}{\varepsilon^2}\left[\rho\left(\frac{u T_s}{1+\varkappa n} + v T_n\right) - \left(\frac{u p_s}{1+\varkappa n} + v p_n\right)\right] = \\ &= \frac{1}{1+\varkappa n}\left[\frac{\mu T_s}{\sigma(1+\varkappa n)}\right]_s + \left[\frac{\mu}{\sigma} T_n\right]_n + \frac{\varkappa}{1+\varkappa n}\,\frac{\mu}{\sigma} T_n + \Phi \end{aligned}$$

$$\Phi = \mu\left[2\left(\frac{u_s + \varkappa v}{1+\varkappa n}\right)^2 + 2v_n^2 + \left(u_n + \frac{v_s - \varkappa u}{1+\varkappa n}\right)^2\right] + \lambda\left[\frac{u_s + \varkappa v}{1+\varkappa n} + v_n\right]^2 \tag{2.4}$$

$$p = \frac{\gamma - 1}{\gamma}\rho T, \quad \mu = \mu(T), \quad \lambda = \lambda(T), \quad \varepsilon = \frac{1}{\sqrt{R}}, \quad R = \frac{V_0 l_0 \rho_0}{\mu_0} \tag{2.5}$$

Here $u$, $v$ are the velocity components in the directions of increasing $s$ and $n$, respectively, $\rho$ the density, $p$ the pressure, $T$ the temperature, $\sigma$ the Prandtl number, $\mu$ and $\lambda$ the coefficients of viscosity; $\gamma$ the adiabatic index, $R$ the Reynolds number formed from the characteristic parameters of the flow; and derivatives are denoted by subscripts, for example, $u_n = \partial u / \partial n$.

In Eqs. (2.1) to (2.5) all lengths are referred to $l_0$, velocities to $V_0$, density to $\rho_0$, pressure to $\rho_0 V_0^2$, temperature to $T_0 = V_0^2 c_p^{-1}$, where $c_p$ is the specific heat of the gas at constant pressure, and coefficients of viscosity to $\mu_0$ ; and the notation for the dimensionless quantities is retained also for dimensional ones. As characteristic parameters $V_0$, $\rho_0$, $\mu_0$, it is convenient in the present case to take the corresponding values of $V$, $\rho$ and $\mu$ in the stream directly ahead of point $O$ (see Fig. 1).

**3.** As is known, for the solution of the problem of flow past a body in the case $\epsilon \ll 1$, one distiguishes the boundary layer — a region of thickness $O(\epsilon)$ directly adjacent to the surface of the body and the "external flow" region.

In the latter region the solution of the system of Eqs. (2.1) to (2.5) is sought in the form of the asymptotic expansion [1]

$$f(s, n, \varepsilon) \sim F_1(s, n) + \varepsilon F_2(s, n) + \ldots \qquad (3.1)$$

Here $f$ stands for $u, v, p, \rho, T$. In the boundary-layer region, where viscous forces are of the same order as inertial forces, a new variable $N = n\epsilon^{-1}$ is introduced, and the solution is sought in the form of the asymptotic expansion

$$f(s, n, \varepsilon) \sim f_1(s, N) + \varepsilon f_2(s, N) + \ldots, \qquad v \sim \varepsilon v_1 + \varepsilon^2 v_2 + \ldots \qquad (3.2)$$

where $f$ stands for $u, p, \rho, T$. The equations for the first terms of the expansion (3.1) are the Euler equations; the equations for the first terms of the expansion (3.2) are the Prandtl equations, which have the form (3.3)

$$(\rho_1 u_1)_s + (\rho_1 v_1)_N = 0, \qquad \rho_1 (u_1 u_{1s} + v_1 u_{1N}) - (\mu u_{1N})_N = -p_{1s}, \qquad p_{1N} = 0$$

$$\rho_1 (u_1 t_{1s} + v_1 t_{1N}) - p_{1s} u_1 = (t_{1N}\mu/\sigma)_N + \mu u_1{}^2{}_N, \ \mu = \mu(t_1), \ p_1 = t_1 \rho_1 (\gamma - 1)/\gamma$$

In the derivation of Eqs. (3.3) it is assumed that $\varkappa$ is a bounded function of $s$ and $\epsilon$.

In order that Eqs. (3.3), which are correct to within a factor $1 + O(\epsilon)$, can be used for the determination of the gas motion in a boundary layer for small but finite $\epsilon$, it is necessary to impose a more stringent limitation on $\varkappa$, namely to assume that $\varkappa$ does not appreciably exceed unity.

In this case the asymptotic bounds are evidently reflected in the actual orders of quantities, and Eqs. (3.3) permit the flow in the boundary layer to be calculated with good accuracy up to the vicinity of point $O$ (see Fig. 1), where $\varkappa \gg 1$. Here, although as before we have the asymptotic estimate of error $1 + O(\epsilon)$ in the equations for fixed $\delta$ and $\epsilon \to 0$, for small but finite $\epsilon$ Eqs. (3.3) can become too crude, and it is appropriate to regard the problem as depending upon the parameters $\epsilon$ and $\delta$. The purpose of the following is to obtain equations for the motion of the gas in the vicinity of point $O$ under the conditions that $\epsilon \to 0$, $\delta \to 0$, that are correct to within a factor $1 + O(\epsilon)$ and can be used for finite $\epsilon$ and $\delta$. It is assumed that the flow outside the boundary layer is supersonic.

For the construction of an asymptotic theory of gas motion in the vicinity of the point $O$ it is necessary to distinguish the following cases

$$\lim \varepsilon\delta^{-1} = 0; \quad \lim \varepsilon\delta^{-1} = \beta_0 < \infty; \quad \lim \varepsilon^2\delta^{-1} = 0, \quad \varepsilon, \delta \to 0$$

$$\lim \varepsilon\delta^{-1} = \infty; \quad \lim \varepsilon^2\delta^{-1} = \beta_1 < \infty; \quad \lim \varepsilon^2\delta^{-1} = \infty, \quad \varepsilon, \delta \to 0$$

We note that the second case was partially considered in [2].

**4.** We consider the case $\lim \epsilon\delta^{-1} = 0$ for $\epsilon, \delta \to 0$. Since in passing from the point on the profile determined by the coordinate $S = s\delta^{-1} = -1$ to the point $S = +1$ the tangent to the profile rotates through a finite angle (cf. conditions (1.1) and (1.2)), the inviscid flow corresponding to the first term of the expansion (3.1) undergoes a finite change in the flow quantities in this segment, in particular of the pressure $p$. In the boundary layer $p_N = 0$ ahead of and behind some neighborhood of point $O$, so that also in a layer of thickness $O(\epsilon)$ in the vicinity of point $O$ the quantity $p = O(1)$ changes by a finite amount; that is

$$\frac{\partial p}{\partial s} = O(\delta^{-1}), \quad \text{or} \quad \frac{\partial p}{\partial S} = O(1), \quad -1 < S < 1$$

For the other gas parameters we assume that differentiation with respect to $S = s\delta^{-1}$ for $-1 < S < 1$ likewise does not change the order of functions for $\epsilon \to 0$, $\delta \to 0$. The gas flow in the boundary layer ahead of point $O$ is a flow with strong shear. Certainly the flow has the same character also in the vicinity of point $O$. We therefore assume that in the vicinity of point $O$, just ahead of it, differentiation of functions with respect to $N = n\epsilon^{-1}$ does not change the order of the functions for $N = O(1)$ and $-1 < S < 1$. We transform Eqs. (2.1) to (2.4) to the variables $S = s\delta^{-1}$ and $N = n\epsilon^{-1}$. The continuity equation (2.1) is written in the form

$$(\rho u)_S + [(1 + K\varepsilon\delta^{-1}N)\rho v\delta\varepsilon^{-1}]_N = 0 \tag{4.1}$$

Since $\delta\epsilon^{-1} \to \infty$ as $\delta \to 0$ and $\epsilon \to 0$, $v = O(\epsilon\delta^{-1})$. We introduce $v^* = O(1)$ by the substitution

$$v = \varepsilon\delta^{-1}v^* \tag{4.2}$$

Eq. (4.1) takes the form

$$(\rho u)_S + [(1 + K\varepsilon\delta^{-1}N)\,\rho v^*]_N = 0 \tag{4.3}$$

Eq. (2.2) in the variables $S$ and $N$, and with consideration of (4.2), is written in the form

$$\begin{gathered}
\rho\left[\frac{uu_S}{1 + KN\varepsilon\delta^{-1}} + v^*u_N + \frac{K}{1 + KN\varepsilon\delta^{-1}}\,uv^*\varepsilon\delta^{-1}\right] + \frac{p_S}{1 + KN\varepsilon\delta^{-1}} = \\
= \delta(\mu u_N)_N + \varepsilon\left\{\left(\mu\,\frac{v_S\,\varepsilon\delta^{-1} - Ku}{1 + KN\varepsilon\delta^{-1}}\right)_N + \varepsilon\delta^{-1}\frac{2}{1 + KN\varepsilon\delta^{-1}}\left[\mu\left(\frac{u_S + Kv^*\varepsilon\delta^{-1}}{1 + KN\varepsilon\delta^{-1}}\right)\right]_S + \right. \\
+ \mu\,\frac{2K}{1 + KN\varepsilon\delta^{-1}}\left(u_N + \frac{v_S^*\varepsilon\delta^{-1} - Ku}{1 + KN\varepsilon\delta^{-1}}\right) + \frac{\varepsilon\delta^{-1}}{1 + KN\varepsilon\delta^{-1}} \times \\
\left.\times\left[\lambda\left(\frac{u_S + Kv^*\varepsilon\delta^{-1}}{1 + KN\varepsilon\delta^{-1}} + v^S\right)\right]_S\right\}
\end{gathered} \tag{4.4}$$

From (4.4) follows Eq.

$$\rho\left(\frac{uu_S}{1 + KN\varepsilon\delta^{-1}} + v^*u_N + \frac{Kuv^*}{1 + KN\varepsilon\delta^{-1}}\,\varepsilon\delta^{-1}\right) + \frac{p_S}{1 + KN\varepsilon\delta^{-1}} = \delta(\mu u_N)_N + O(\varepsilon) \tag{4.5}$$

The equations of gas motion are being formed with uniform accuracy $1 + O(\epsilon)$, so that in Eqs. (4.3) and (4.5) terms of the form $KN\epsilon\delta^{-1}$, $\delta(\mu u_N)_N$ must be retained. After analogous transformation, Esq. (2.3) and (2.4) are written as

$$p_N - \frac{K\rho u^2}{1 + KN\varepsilon\delta^{-1}}\,\varepsilon\delta^{-1} + \rho\,(\varepsilon\delta^{-1})^2\left(\frac{uv_S^*}{1 + KN\varepsilon\delta^{-1}} + v^*v_N^*\right) = O(\varepsilon^2\delta^{-1}) \tag{4.6}$$

$$\begin{gathered}
\rho\left(\frac{uT_S}{1 + KN\varepsilon\delta^{-1}} + v^*T_N\right) - \left(\frac{up_S}{1 + KN\varepsilon\delta^{-1}} + v^*p_N\right) = \\
= \delta\left[\left(\frac{\mu}{\sigma}\,T_N\right)_N + \mu u_N^2\right] + O(\varepsilon)
\end{gathered} \tag{4.7}$$

On the surface of the body $N = 0$, $u = v^* = 0$ (no-slip condition), and Eq.(4.5) reduces to

$$p_S = \delta(\mu u_N)_N + O(\varepsilon) \tag{4.8}$$

The left-hand side of (4.8) contains a finite quantity, and the right-hand side is infinitesimally small for $\varepsilon \to 0$ and $\delta \to 0$, so that near the wall there is a layer where derivatives of functions with respect to $N$ are of different order than the functions themselves. It follows from (4.8) that in this layer it is necessary to introduce the variable

$$\eta = N\delta^{-1/2} = n\varepsilon^{-1}\delta^{-1/2} \tag{4.9}$$

The continuity equation (2.1) shows that $v = O(\varepsilon\delta^{-\frac{1}{2}})$ for $\eta = O(1)$. With the substitution $v = \varepsilon\delta^{-\frac{1}{2}}v^\circ$ and transformation to the variables $S$ and $\eta$, Eqs.(2.1) to (2.4) assume the form

$$(\rho u)_S + [(1 + K\eta\varepsilon\delta^{-1/2})\rho v^\circ]_\eta = 0 \tag{4.10}$$

$$\rho\left(\frac{uu_S}{1 + K\eta\varepsilon\delta^{-1/2}} + v^\circ u_\eta + \frac{Kuv^\circ}{1 + K\eta\varepsilon\delta^{-1/2}}\right) + \frac{p_S}{1 + K\eta\varepsilon\delta^{-1/2}} = \tag{4.11}$$

$$= (\mu u_\eta)_\eta + \varepsilon\delta^{-1/2}[-(\mu K u)_\eta + 2K\mu u_\eta] + O(\varepsilon^2\delta^{-1})$$

$$p_\eta - K\rho u^2\varepsilon\delta^{-1/2} = O(\varepsilon^2\delta^{-1}) \tag{4.12}$$

$$\rho\left(\frac{uT_S}{1 + K\eta\varepsilon\delta^{-1/2}} + v^\circ T_\eta\right) - \left(\frac{up_S}{1 + K\eta\varepsilon\delta^{-1/2}} + v^\circ p_\eta\right) = \tag{4.13}$$

$$= \left(\frac{\mu}{\sigma} T_\eta\right)_\eta + \mu u_\eta^2 + \varepsilon\delta^{-1/2}K\mu(\sigma^{-1}T_\eta - 2uu_\eta) + O(\varepsilon^2\delta^{-1})$$

**5.** We consider the case $\lim \varepsilon\delta^{-1} = \beta_0 < \infty$ for $\varepsilon, \delta \to 0$. We represent $\delta$ in the form of the product $\varepsilon\beta^{-1}$, where $\beta = O(1)$ when $\varepsilon, \delta \to 0$, and substitute $\delta = \varepsilon\beta^{-1}$ into Eqs. (4.2) to (4.13).

As a result we obtain the equations of gas motion in the form

$$(\rho u)_S + [(1 + KN\beta)\rho v^*]_N = 0,$$

$$\rho\left(\frac{uu_S}{1 + KN\beta} + v^* u_N + \frac{Kuv^*\beta}{1 + KN\beta}\right) + \frac{p_S}{1 + KN\beta} = O(\varepsilon)$$

$$\rho\left(\frac{\beta^2 uv_S^*}{1 + kN\beta} + \beta^2 v^* v_N^* - \frac{K\beta u^2}{1 + KN\beta}\right) + p_N = O(\varepsilon) \tag{5.1}$$

$$\rho\left(\frac{uT_S}{1 + KN\beta} + v^* T_N\right) - \left(\frac{up_S}{1 + KN\beta} + v^* p_N\right) = O(\varepsilon)$$

$$(v = \beta v^*,\ N = n\varepsilon^{-1},\ S = \beta s\varepsilon^{-1})$$

$$(\rho u)_S + [(1 + K\beta^{1/2}\varepsilon^{1/2})\rho v^\circ]_\eta = 0$$

$$\rho\left[\frac{uu_S}{1 + K\eta(\beta\varepsilon)^{1/2}} + v^\circ u_\eta + Kuv^\circ(\varepsilon\beta)^{1/2}\right] + \frac{p_S}{1 + K\eta(\varepsilon\beta)^{1/2}} =$$

$$= (\mu u_\eta)_\eta + (\beta\varepsilon)^{1/2}[-(\mu K u)_\eta + 2K\mu u_\eta] + O(\varepsilon)$$

$$P_\eta - K\rho u^2(\beta\varepsilon)^{1/2} = O(\varepsilon) \tag{5.2}$$

$$\rho\left[\frac{uT_S}{1 + K\eta(\beta\varepsilon)^{1/2}} + v^\circ T_\eta\right] - \left[\frac{up_S}{1 + K\eta(\beta\varepsilon)^{1/2}} + v^\circ p_\eta\right] =$$

$$= \left(\frac{\mu}{\sigma} T_\eta\right)_\eta + \mu u_\eta^2 + (\beta\varepsilon)^{1/2} K\mu (\sigma^{-1} T_\eta - 2uu_\eta) + O(\varepsilon)$$

$$(v = (\beta\varepsilon)^{1/2} v^\circ, \quad \eta = n\beta^{1/2}\varepsilon^{-3/2}, \quad S = \beta s\varepsilon^{-1})$$

**6.** We consider the case $\lim \epsilon^2\delta^{-1} = 0$ but $\lim \epsilon\delta^{-1} = \infty$ for $\epsilon$, $\delta \to 0$. The continuity equation (2.1) is written in the variables $S$ and $N$ in the form

$$(\rho u)_S + [(\delta\varepsilon^{-1} + KN)\rho v]_N = 0 \tag{6.1}$$

Hence it follows that $v = O(1)$ for $\epsilon$, $\delta \to 0$. Eq. (2.2), ... takes the form

$$\rho\left(\frac{uu_S}{KN + \delta\varepsilon^{-1}} + vu_N + \frac{Kuv}{KN + \delta\varepsilon^{-1}}\right) + \frac{p_S}{KN + \delta\varepsilon^{-1}} = O(\varepsilon), \ldots \tag{6.2}$$

In the variables $\eta$ and $S$, after the substitution $v = \epsilon\delta^{-\frac{1}{2}} v^\circ$, Eqs. (2.1), (2.2), etc. take the form

$$(\rho u)_S + [(1 + K\eta\varepsilon\delta^{-1/2})\rho v^\circ]_\eta = 0 \tag{6.3}$$

$$\rho\left(\frac{uu_S}{1 + K\eta\varepsilon\delta^{-1/2}} + v^\circ u_\eta + \frac{Kuv^\circ}{1 + K\eta\varepsilon\delta^{-1/2}}\varepsilon\delta^{-1/2}\right) + \frac{p_S}{1 + K\eta\varepsilon\delta^{-1/2}} =$$

$$= (\mu u_\eta)_\eta + \left[\mu\left(\frac{\varepsilon\delta^{-1/2} v^\circ - Ku}{1 + K\eta\varepsilon\delta^{-1/2}}\right)\right]_\eta \varepsilon\delta^{-1/2} + \varepsilon^2\delta^{-1}\left[\mu\left(\frac{u_S + Kv^\circ\varepsilon\delta^{-1/2}}{1 + K\eta\varepsilon\delta^{-1/2}}\right)\right]_S + \tag{6.4}$$

$$+ \mu\frac{2K}{1 + K\eta\varepsilon\delta^{-1/2}}\left(u_\eta\varepsilon\delta^{-1/2} + \frac{v_S^\circ\varepsilon\delta^{-1/2} - Ku\varepsilon^2\delta^{-1}}{1 + K\eta\varepsilon\delta^{-1/2}}\right) +$$

$$+ \frac{\varepsilon^2\delta^{-1}}{1 + K\eta\varepsilon\delta^{-1}}\left[\lambda\left(\frac{u_S + Kv^\circ\varepsilon\delta^{-1/2}}{1 + K\eta\varepsilon\delta^{-1/2}} + v_\eta^\circ\right)\right]_S, \ldots$$

Since the problem is to obtain equations of motion with an error bound in the form of the factor $1 + O(\epsilon)$, it is necessary in Eq. (6.4) to retain terms $O(\epsilon^2\delta^{-1})$, since $\lim(\epsilon^2\delta^{-1}/\epsilon) = \infty$ for $\epsilon$, $\delta \to \infty$; that is, it is necessary to solve the full Navier-Stokes equations. Eqs. (2.1) to (2.4) can nevertheless be reduced to a system of equations of boundary-layer type, but with an error estimate in the form of the factor $1 + O(\epsilon^2\delta^{-1})$.

**7.** We consider the case $\lim \epsilon^2\delta^{-1} = \beta_1 < \infty$ for $\epsilon$, $\delta \to 0$. If $\delta$ is represented by the product $\epsilon^2\beta^{-1}$, where $\beta = O(1)$ when $\delta$, $\epsilon \to 0$, then after substitution of $\delta = \epsilon^2\beta^{-1}$ into Eqs. (6.3), (6.4) and the others it is easy to see that the equations of gas motion are the full Navier-Stokes equations; that is, it is not possible to reduce the equations of the problem to a parabolic system of equations of boundary-layer type if $\delta = O(\epsilon^2)$ (or still higher order).

**8.** When specific values of $\epsilon$ and $\delta$ are given, it is necessary first to decide with which asymptotic case the data are associated, and then how to integrate the boundary-layer equations. (Values of $\epsilon$ of practical interest lie in the range $10^{-3} < \epsilon < 10^{-1}$; no bounds can be imposed upon $\delta$.) Here different situations can arise depending upon the values of $\epsilon$ and $\delta$. We consider some typical cases as examples.

**9.** Let $\varepsilon = 2\cdot 10^{-3}$, $\delta = 10^{-1}$. Then $\beta = \varepsilon\delta^{-1} = 2\cdot 10^{-2}$ and it follows that one should use the equations for the case $\lim \varepsilon\delta^{-1} = 0$ for $\varepsilon, \delta \to 0$. Because $\varepsilon\delta^{-1/2} = 64\cdot 10^{-4}$, and $(\varepsilon\delta^{-1})^2 = 4\cdot 10^{-4}$, we may in Eqs. (4.3), (4.5) to (4.7) and (4.10) to (4.13) omit containing factors $\varepsilon\delta^{-1/2}$ and $(\varepsilon\delta^{-1})^2$, as being of the order of $\epsilon$.

Because Eqs. (4.3) and (4.5) to (4.7) contain the same terms as the corresponding Eqs. (4.10) to (4.13) in the given specific case, it is appropriate to use the system of Eqs.

(4.3) and (4.5) to (4.7) up to $N=0$, and solve them by finite-difference methods, taking the steps in $N$ three times smaller for $N<0.5$ than for $N>0.5$ (approximately). Correction of the pressure change in its dependence upon $N$ can be introduced in an iterative way. If terms $O(\varepsilon\delta^{-\frac{1}{2}})$ are retained in the equations, it is appropriate to use Eqs. (4.3) and (4.5) to (4.7) as before for all values of $N$, but these equations must be supplemented by terms that are small for $N=O(1)$ but not small for $\eta=O(1)$; for example, in the right-hand side of Eq. (4.7) one must retain a term $\varepsilon K\mu(\sigma^{-1}T_N-2uu_N)$, corresponding to the term $\varepsilon\delta^{-1/2}K\mu(\sigma^{-1}T_\eta-2uu_\eta)$ in Eq. (4.13), that is, construct a hybrid equation, etc.

**10.** Let $\varepsilon=2\cdot10^{-3}$, $\delta=10^{-2}$. Then $\beta=\varepsilon\delta^{-1}=2\cdot10^{-1}$, and it follows that one should use the equations for the case $\lim\varepsilon\delta^{-1}=0$ for $\varepsilon$, $\delta\to0$. Because $\varepsilon\delta^{-1/2}=2\cdot10^{-2}$, and $(\varepsilon\delta^{-1})^2=4\cdot10^{-2}$, it is necessary to retain all terms written out in the equations. Since $\varepsilon$ and $\delta$ are rather small, it would seem natural to seek the solution of the problem in the form: for $N=O(1)$

$$f(S,N,\varepsilon,\delta)=F_0(S,N,\varepsilon/\delta)+\delta^{1/2}F_{1/2}(S,N,\varepsilon/\delta)+$$
$$+\delta F_1(S,N,\varepsilon/\delta)+O(\delta^{3/2})+O(\varepsilon^2\delta^{-1})\tag{10.1}$$

for $\eta=O(1)$

$$f(S,\eta,\varepsilon,\delta)=f_0(S,\eta)+\varepsilon\delta^{-1/2}f_1(S,\eta)+O(\varepsilon^2\delta^{-1})\tag{10.2}$$

Here $f=p$, $\rho$, $T$, $u$, $v^*$ or $v^\circ$. But to attain a uniformly accurate solution it would be necessary to take three terms in Eq. (10.1) and two in Eq. (10.2) (since $\varepsilon=2\cdot10^{-3}$, $\delta^{3/2}=10^{-3}$, $\varepsilon\delta^{-1/2}=2\cdot10^{-2}$), the determination of which represents a difficult problem. For this reason it is simpler to use the equations in the $S$, $N$ variables, in which in addition are included terms that are small for $N=O(1)$ but not small for $\eta=O(1)$, as was done in the previous example, and solve these equations by finite-difference methods for $N\geq0$, $S>S_0$ (where $S_0$ is a value of order $-1$). In so doing it is necessary to consider that the step in $N$ should be taken smaller by a factor $\sqrt{\delta}=10$ (approximately) for $N<0.5$ than for $N>0.5$. If one considers two terms in the expansion (10.1) and one in (10.2), the error in the solution will be $O(\delta)$ (that is, an order larger than in the boundary layer ahead of point $O$).

**11.** Let $\varepsilon=2\cdot10^{-2}$, $\delta=10^{-2}$. Then $\beta=\varepsilon\delta^{-1}=2$, and it follows that one should use the equations for the case $\lim\varepsilon\delta^{-1}=\beta_0$ for $\varepsilon,\delta\to0$. For $N=O(1)$ the gas motion is described by the Euler Eqs. (5.1) to an accuracy of $1+O(\varepsilon)$, and for $\eta=O(1)$ by the boundary-layer equations (5.2) with the same accuracy. The solution of the problem with uniform accuracy has the form

$$f(S,\eta,\varepsilon)=f_0(S,\eta)+(\beta\varepsilon)^{1/2}f_1(S,\eta)+O(\varepsilon)\quad\text{for}\quad\eta=O(1)\tag{11.1}$$
$$f(S,N,\varepsilon)=F_0(S,N)+(\beta\varepsilon)^{1/2}F_1(S,N)+O(\varepsilon)\quad\text{for}\quad N=O(1)\tag{11.2}$$

where $f=p,\rho,T,u,v^*$ or $f=v^\circ$. The appearance of terms $O(\varepsilon^{\frac{1}{2}})$ in (11.2) is connected with the effect of the "displacement thickness" of the boundary layer, in which

$$v=(\beta\varepsilon)^{1/2}v^\circ=(\beta\varepsilon)^{1/2}[v_0^\circ(S,\eta)+(\beta\varepsilon)^{1/2}v_1^\circ(S,\eta)+O(\varepsilon)]$$

For large $\eta$ ($\eta\to\infty$)

$$v_0^\circ(S,\eta)=A_1(S)\eta+A_0(S)+o(1)$$

Here $A_1(S)$ and $A_0(S)$ are functions of $S$; $A_0(S)$ determines the "displacement thickness" of the boundary layer. The problem of finding the first two terms in the expansions (11.1) and (11.2) resembles the problem of finding the second approximation in boundary-layer theory, the solution of which was given by M. Van Dyke in [2], but here the parameter with which the expansions are formed is $(\beta\epsilon)^{\frac{1}{2}}$ and not $\epsilon$. For this reason we devote our main attention to the differences occurring in this problem. It is not cleer whether the regions where the asymptotic expansions (3.2) and (11.2) are valid will overlap. It therefore evidently follows that Eqs. (3.3) should be used up to some value $S = S_0$ (of order $-1$), and then for $S > S_0$ one should change to a hybrid system of equations containing all terms of Eqs. (3.3) as well as all terms of Eqs. (5.1) and (5.2). After transition through the speed of sound outside a layer $O(\varepsilon^{3/2})$ at $S = S_0{}^*$ we may use the systems (5.1) and (5.2) separately (another approach to the solution of these equations is given in [2]). For $S > S_0{}^*$ the problem is solved analogously to the problem of the second approximation in [1].

**12.** Let $\epsilon = 10^{-3}$, $\delta = 10^{-3}$. Since $\beta = \epsilon\delta^{-1} = 1$, it follows that one should use the equations for the case $\lim \epsilon\delta^{-1} = \beta_0$ for $\epsilon$, $\delta \to 0$ and the method of solution considered in the previous example. However, there is the possibility here of taking one term in the expansions (11.1) and (11.2). As a result, the error in the solution of the problem will be $O(\epsilon^{\frac{1}{2}})$ rather than $O(\epsilon)$, that is, hundredths and not thousandths, which may, however, be sufficient for a rough calculation. We note that the error in the solution for $S > 1$ will also be $O(\epsilon^{\frac{1}{2}})$.

**13.** Let $\varepsilon = 3 \cdot 10^{-2}$, $\delta = 10^{-3}$. Since $\varepsilon^2\delta^{-1} = 0.9$, the problem cannot be solved by the methods of boundary-layer theory.

**14.** After determination of the solution of the problem in the vicinity of point $O$, for $S > S_1$, where $S_1$ is a quantity of order 1, one should change to the system of Eqs. (3.3). As a result, the solution of the problem of the plane laminar boundary layer will be obtained with uniform accuracy by a factor $1 + O(\epsilon)$, provided that the flow does not separate in the vicinity of point $O$.

## BIBLIOGRAPHY

1. Neiland, V. Ia. and Sychev, V. V., Asymptotic solutions of the Navier-Stokes equations in regions with large local disturbances. Izv. Akad. Nauk SSSR, Mekh. Zhid. Gaza, No. 4, pp. 43-49, 1966.
2. Van Dyke, M., Second-order compressible boundary-layer theory with application to blunt bodies in hypersonic flow. In Hypersonic Flow Research, New York-London, Academic Press, pp. 37-76, 1962.

Translated by M. D. v. D.